

# Properties of the solutions of the conjugate heat equations

Richard Hamilton, Natasa Sesum

## Abstract

In this paper we consider the class  $\mathcal{A}$  of those solutions  $u(x, t)$  to the conjugate heat equation  $\frac{d}{dt}u = -\Delta u + Ru$  on compact Kähler manifolds  $M$  with  $c_1 > 0$  (where  $g(t)$  changes by the unnormalized Kähler Ricci flow, blowing up at  $T < \infty$ ), which satisfy Perelman's differential Harnack inequality on  $[0, T)$ . We show  $\mathcal{A}$  is nonempty. If  $|\text{Ric}(g(t))| \leq \frac{C}{T-t}$ , which is always true if we have type I singularity, we prove the solution  $u(x, t)$  satisfies the elliptic type Harnack inequality, with the constants that are uniform in time. If the flow  $g(t)$  has a type I singularity at  $T$ , then  $\mathcal{A}$  has exactly one element.

## 1 Introduction

Let  $M$  be a Kähler manifold with  $c_1(M) > 0$ , of complex dimension  $n$ . Consider the solutions to the unnormalized Kähler Ricci flow,

$$\frac{d}{dt}g_{i\bar{j}} = -R_{i\bar{j}}. \quad (1)$$

It is known in the case of the unnormalized Kähler Ricci flow that it shrinks to a point, after some finite time  $T < \infty$ . Let  $T' < T$  and let  $u = (4\pi(T' - t))^{-n}e^{-f}$  satisfy the conjugate heat equation

$$\frac{d}{dt}u = -\Delta u + Ru. \quad (2)$$

This implies  $f$  satisfies,

$$\frac{d}{dt}f = -\Delta f + |\nabla f|^2 - R + \frac{n}{T-t}. \quad (3)$$

Let

$$v = [(T' - t)(2\Delta f - |\nabla f|^2 + R) + f - 2n]u, \quad (4)$$

which is such that  $\int_M v$  is exactly Perelman's functional  $\mathcal{W}$ . He proved it is monotonically increasing along the flow, that is,

$$\frac{d}{dt}\mathcal{W} = 2\tau \int_M |R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f - g_{i\bar{j}}|^2 u dV \geq 0,$$

If  $u$  tends to a  $\delta$ -function as  $t \rightarrow T'$ , in [6], Perelman proved  $v \leq 0$  for all  $t \in [0, T']$ . He also proved that under the same assumptions as above, for any smooth curve  $\gamma(t)$  in  $M$ ,

$$-\frac{d}{dt}f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(T' - t)}f(\gamma(t), t), \quad (5)$$

for all  $t \in [0, T']$ .

**Definition 1.** We will say that a smooth function  $f$  is *admissible* if for any smooth curve  $\gamma(t)$  in  $M$ ,

$$-\frac{d}{dt}f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(T - t)}f(\gamma(t), t), \quad (6)$$

for all  $t \in [0, T)$ , while the metric changes by the Ricci flow equation (1) and  $T$  is a time at which the flow disappears.

We will prove the following results about  $u$ .

**Theorem 2.** *If  $|\text{Ric}(g(t))| \leq \frac{c}{T-t}$ , which translates to the condition,  $\text{Ric} \geq -c$  along the normalized Kähler Ricci flow, the set  $\mathcal{A}$  is nonempty and there is a uniform constant  $C$ , so that,*

$$\max_{M \times [0, T)} u(x, t) \leq C \min_{M \times [0, T)} u(x, t).$$

*If we assume the flow has a type I singularity, meaning that  $|\text{Rm}(g(t))| \leq \frac{C}{T-t}$ , there is exactly one element in  $\mathcal{A}$ , that is, the solution to the conjugate heat equation (2), existing all the way up to  $T$  and satisfying (6) is unique.*

The organization of the paper is as follows. In section 2 we will give the proof of Theorem 2. Complex two dimensional case will be discussed in section 3. In section 4 we will discuss Perelman's reduced distance function and show how its definition can be extended to a distance function with a base point at  $(p, T)$ , where  $p$  is the point to which the flow shrinks and  $T$  is the singular time.

## 2 Harnack type estimates and the uniqueness of $u$

We will assume for the moment that  $\mathcal{A}$  is not empty and prove that each element  $u(t) \in \mathcal{A}$  satisfies the elliptic Harnack inequality at each time slice with the uniform constant, not depending on time, and that such a solution is unique if  $g(t)$  has type I singularity at  $T$ .

**Proposition 3.** *If  $|\text{Ric}(g(t))| \leq \frac{C}{T-t}$  along the flow  $g(t)$ , there exists a uniform constant  $\tilde{C}$ , so that*

$$\max_M u(x, t) \leq \tilde{C} \min_M u(x, t),$$

for all  $t \in [0, T)$ .

*Proof.* Take  $t_1 < t_2 < T$  and  $x_1, x_2 \in M$ . Let  $\gamma(t)$  be a curve that will be chosen later, so that it connects  $x_1$  and  $x_2$ , that is,  $\gamma(t_1) = x_1$  and  $\gamma(t_2) = x_2$ . Since  $f$  is an admissible function, it satisfies (6), for a chosen curve  $\gamma$ . Integrate it in  $t \in [t_1, t_2]$ . We get,

$$f(x_1, t_1)\sqrt{T-t_1} \leq f(x_2, t_2)\sqrt{T-t_2} + \frac{1}{2} \int_{t_1}^{t_2} \sqrt{T-t} (R(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt. \quad (7)$$

By translation in time, we may assume  $T = 1/2$ . It easily follows that if we rescale the flow, that is, if  $\tilde{g}(s(t)) = \frac{g(t)}{1-t/T} = Tg(t)/(T-t)$  with  $s(t) = -T \ln(1-t/T)$ , we get a normalized Kähler Ricci flow, satisfying,

$$\frac{d}{ds} \tilde{g} = \tilde{g} - \text{Ric}(\tilde{g}),$$

for all  $s \in [0, \infty)$ . By Perelman's results,  $|R(\tilde{g}(s))| \leq C$  and  $\text{diam}(M, \tilde{g}(s)) \leq C$  along the flow. This implies

$$R(g(t)) \leq \frac{C}{T-t}, \quad (8)$$

and

$$\text{diam}(M, g(t)) \leq C\sqrt{T-t}. \quad (9)$$

As a matter of reparametrization, we also get

$$\text{Vol}_{g(t)} = (1-t/T)^n \text{Vol}_{\tilde{g}(s(t))}(M) = C(T-t)^n. \quad (10)$$

We will estimate the integral term appearing in (7). By (8) we have,

$$\int_{t_1}^{t_2} \sqrt{T-t} R(\gamma(t), t) dt \leq C \int_{t_1}^{t_2} \frac{dt}{\sqrt{T-t}} = C \frac{t_2 - t_1}{\sqrt{T-t_1} + \sqrt{T-t_2}}. \quad (11)$$

Without loosing a generality assume that  $|\text{Ric}(g(t))|(T-t) \leq 1$ , which may be always achieved by rescaling. We have the simple claim that follows immediatelly from the evolution equation for  $g(t)$ .

**Claim 4.** *If  $\text{Ric}(g(t))(T-t) \geq -g(t)$  for all  $t \in [0, T)$ , for any  $0 \leq t < s < T$ , we have,*

$$g(s) \leq \frac{T-t}{T-s}g(t).$$

*In particular, for any vector  $v$ , we have*

$$|v|_{g(s)}^2 \leq \frac{T-t}{T-s}|v|_{g(t)}^2.$$

Let  $s_1 = s(t_1)$ . If we choose  $\gamma$  to be a minimal geodesic from  $x_1$  to  $x_2$  with respect to  $g(t_1)$ , by (9) and by the previous claim, we have,

$$\begin{aligned} \int_{t_1}^{t_2} \sqrt{T-t} |\dot{\gamma}|^2 dt &= \int_{T-t_2}^{T-t_1} \sqrt{\tau} |\dot{\gamma}|_{g(T-\tau)}^2 d\tau \\ &\stackrel{\sqrt{\tau}=s}{=} \int_{\sqrt{T-t_2}}^{\sqrt{T-t_1}} \frac{1}{2} |\gamma'|_{g(T-s^2)}^2 ds \\ &\leq C \int_{\sqrt{T-t_2}}^{\sqrt{T-t_1}} \frac{T-t_1}{T-t_2} |\gamma'|_{g(t_1)}^2 ds \\ &= \tilde{C} \frac{T-t_1}{T-t_2} \frac{\text{dist}_{g(t_1)}^2(x_1, x_2)}{\sqrt{T-t_1} - \sqrt{T-t_2}} \\ &\stackrel{(9)}{\leq} \bar{C} \frac{(T-t_1)^2 \sqrt{T-t_1}}{(T-t_2)(t_2-t_1)}. \end{aligned} \tag{12}$$

From estimates (7), (11) and (12), we get

$$f(x_1, t_1) \leq \frac{\sqrt{T-t_2}}{\sqrt{T-t_1}} f(x_2, t_2) + C \frac{t_2-t_1}{T-t_1} + C \frac{(T-t_1)^2}{(t_2-t_1)(T-t_2)}.$$

If (\*)  $t_2 - t_1 \sim T - t_1$ , e.g.  $t_2 - t_1 = T - t_2 = \delta$ , then,

$$f(x_1, t_1) \leq \frac{1}{\sqrt{2}} f(x_2, t_2) + C, \tag{13}$$

and since  $x_1, x_2$  were two arbitrary points, we have

$$\max_M f(\cdot, t_1) \leq \frac{1}{\sqrt{2}} \min_M f(\cdot, t_2) + C. \tag{14}$$

We claim there is some  $\tilde{A}$  so that  $f(x, t) \geq -\tilde{A}$ , for all  $x \in M$  and all  $t \in [0, T)$ . Assume  $\max_M f(\cdot, t) \leq -A$ . That implies  $f(x, t) \leq -A$ , for all  $x \in M$  and therefore,

$$(4\pi\tau(T-t))^n = \int_M e^{-f} dV_t \geq e^A \text{Vol}_t(M) = e^A C(T-t)^n,$$

for a uniform constant  $C$ , which is not possible for big enough  $A$  (notice that the bigness of  $A$  does not depend on  $t \in [0, T)$ ). Estimate (14) now implies

$$\min_M f(\cdot, t_2) \geq -\tilde{A},$$

for a uniform constant  $\tilde{A}$ , independent of  $t_2$ . Rewrite (13) as

$$\begin{aligned} \max_M f(\cdot, t_1) &\leq \frac{1}{\sqrt{2}}(\min_M f(\cdot, t_2) + \tilde{A}) - \frac{\tilde{A}}{\sqrt{2}} + C \\ &\leq \min_M f(\cdot, t_2) + \tilde{A} + C_1 = \min_M f(\cdot, t_2) + C_2. \end{aligned}$$

If we denote by  $M(t) = \max_M u(\cdot, t)$  and by  $m(t) = \min_M u(\cdot, t)$ , this yields

$$M(t_2) \leq C\left(\frac{T-t_1}{T-t_2}\right)^n m(t_1) \leq \tilde{C}m(t_1). \quad (15)$$

The evolution equation  $\frac{d}{dt}u = -\Delta u + Ru$  at the points where  $u(\cdot, t)$  achieves its maximum becomes,

$$\frac{d}{dt}M(t) \geq RM(t) \geq -\frac{C}{T-t}M(t),$$

which yields  $M(t_2) \geq \left(\frac{T-t_2}{T-t_1}\right)^C M(t_1) = \tilde{C}M(t_1)$ . This together with (15) gives,

$$M(t_1) \leq C m(t_1), \quad (16)$$

for a uniform constant  $C$  and all  $t_1 \in [0, T)$ , which is an analogue of the Harnack inequality that we have in the elliptic case.  $\square$

We will now prove the nonemptiness of  $\mathcal{A}$ .

**Lemma 5.** *For every Kähler manifold  $M$  and every unnormalized Kähler Ricci flow  $g(t)$ ,  $|\mathcal{A}| \geq 1$ .*

*Proof.* For every unnormalized Kähler Ricci flow  $g(t)$  there is a finite time  $T$  at which the flow disappears. Take an arbitrary increasing sequence of times  $t_i \uparrow T$  and a sequence of points  $p_i \in M$ . For every  $i$ , let  $u_i(t) =$

$(4\pi(t_i - t))^{-n}e^{-f_i(t)}$  be a solution to the conjugate heat equation (2), such that  $u_i(t)$  converges to a  $\delta$ -function as  $t \rightarrow t_i$ , concentrated at  $p_i$ . Let  $v_i(t)$  be a corresponding  $v$ -function as in (4). Due to Perelman (see [6]), we have  $v_i(t) \leq 0$  for all  $t \in [0, t_i]$  and for every smooth curve  $\gamma(t)$ ,

$$-\frac{d}{dt}f_i(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(t_i - t)}f_i(\gamma(t), t),$$

holds for all  $t \in [0, t_i]$ .

Fix  $t_j$  from the sequence of times and consider  $u_i(t)$  for  $i \geq j$ . Then the metrics  $\{g(t)\}$  are uniformly equivalent and their geometries are uniformly bounded for  $t \in [0, t_j]$ , with bounds depending on  $g(t)|_{[0, t_j]}$ . Each  $u_i(t)$  satisfies the conjugate heat equation (2), with  $\int_M u_i(t) dV_t = 1$ . Moreover,

$$\frac{d}{dt}f_i = -\Delta f_i + |\nabla f_i|^2 - R + \frac{n}{t_i - t},$$

which implies,

$$\frac{d}{dt}(f_i)_{\min} \leq \frac{C}{T - t} + \frac{n}{t_i - t} \leq \frac{\tilde{C}}{t_i - t},$$

and therefore, for  $t \in [0, t_j]$ ,

$$(f_i)_{\min}(t_j) \leq (f_i)_{\min}(t) + \tilde{C} \ln\left(\frac{t_i - t}{t_i - t_j}\right).$$

This yields,

$$(f_i)_{\min}(t) \geq -C(t_j),$$

for  $i$  big enough, so that  $t_i \geq \frac{T+t_j}{2}$ , and henceforth,

$$\max_{M \times [0, t_j]} u_i(x, t) \leq e^{-C(t_j)} (2\pi(T - t_j))^{-n} = \tilde{C}(t_j). \quad (17)$$

If we denote by  $\tilde{u}_i(x, t) = u_i(x, t)^{1/2}$ , integrating  $v_i(x, t) \leq 0$ , similarly as in [7], by using Hölder and Sobolev inequalities,

$$\int_M |\nabla \tilde{u}_i|^2 dV_t \leq C_j,$$

which together with (17) imply,

$$\sup_{t \in [0, t_j]} \|u_i(t)\|_{W^{1,2}} \leq C_j,$$

for all  $i$  big enough. By standard parabolic estimates applied to  $u_i(t)$  and  $t \in [0, t_j]$ , it follows there exists  $C(k, l, n, t_j)$  so that

$$|\frac{d^l}{dt^l} u_i(t)|_{C^k} \leq C(k, l, n, t_j),$$

for all  $t \in [0, t_j]$  and all  $i$  sufficiently big. Extract a subsequence  $u_i(t, x)$  that converges in  $C^{k, l}(M \times [0, t_j])$  norm to some function  $u(t, x)$ , defined on  $[0, t_j]$  that continues to be a solution to the conjugate heat equation (2). By taking larger and larger  $j$ , diagonalizing our sequence  $u_i(t)$ , taking into account the uniqueness of the limit, we get a function  $u(t, x)$ , defined on  $M \times [0, T)$  and a subsequence  $u_i(t, x)$ , so that  $u_i(t, x) \xrightarrow{C^{k, l}(M \times [0, T'])} u(t, x)$ , for every  $T' < T$ . Moreover,

- $u(t, x)$  satisfies the conjugate heat equation (2) for all  $t \in [0, T)$ .
- $u(t, x) = (4\pi(T - t))^{-n} e^{-f}$ , where  $f$  is an admissible function in the sense of Definition 1.

In particular, this implies  $|\mathcal{A}| \geq 1$  □

We will now prove the uniqueness part of Theorem 2.

**Proposition 6.** *If  $g(t)$  has type I singularity at  $T$ , then  $|\mathcal{A}| = 1$ , that is, the solution  $u$  to the conjugate heat equation  $\frac{d}{dt}u = -\Delta u + Ru$ , that satisfies Perelman's differential Harnack inequality (6) is unique.*

*Proof.* Assume there are at least two different solutions  $u_1$  and  $u_2$ , with the properties as above. By Theorem 2, we have  $M_j(t) \leq C m_j(t)$ , for all  $t \in [0, T)$  and  $j \in \{1, 2\}$ . We will omit the subscript  $j$  below. On the other hand, we have the integral normalization condition  $(4\pi(T - t))^{-n} \int_M u dV_{g(t)} = 1$ . Combining these two facts, we get

$$m(t) \geq \frac{1}{C \text{Vol}_{g(t)}(M)}, \quad (18)$$

$$M(t) \leq C \frac{1}{\text{Vol}_{g(t)}(M)}, \quad (19)$$

where a constant  $C$  comes from (16). Take a sequence  $t_i \rightarrow T$  and consider a sequence of rescaled metrics  $g_i(t) = (T - t_i)^{-1} g((T - t_i)t + t_i)$ , for  $t \in$

$[-(T - t_i)^{-1}t_i, 1)$ . We get

$$\begin{aligned}
|\text{Rm}(g_i(t))| &= |\text{Rm}(g(t(T - t_i) + t_i))|(T - t_i) \\
&= |\text{Rm}(g(t(T - t_i) + t_i))|\frac{(T - t_i)}{T - t_i - t(T - t_i)} \\
&\leq C\frac{1}{1 - t} \leq \tilde{C},
\end{aligned}$$

for all  $t \in [-(T - t_i)^{-1}t_i, 1/2]$ . By Perelman's volume noncollapsing result and by Hamilton's compactness theorem, there is a subsequence  $(M, g_i(t))$ , converging to another solution to the ancient Kähler Ricci solution  $(M, h(t))$ , defined for  $t \in (-\infty, 1/2]$ . If we also rescale our solution  $u$ , together with our metric  $g(t)$ , estimates (18), (19) and (10) give

$$\begin{aligned}
u_i(t) &= (T - t_i)^n u(t(T - t_i) + t_i) \\
&\leq C \frac{(T - t_i)^n}{\text{Vol}_{g(t(T - t_i) + t_i)}(M)} \\
&= \tilde{C} \frac{(T - t_i)^n}{(T - t_i)^n (1 - t)^n} \leq \bar{C},
\end{aligned}$$

for all  $t \in [-(T - t_i)^{-1}t_i, 1/2]$ . Notice that function  $f$  rescales as  $f_i(t) = f(t_i + t(T - t_i))$ . Similarly we get the uniform lower bound on  $u_i(t)$ , that is, there exists a uniform constant  $\tilde{C} > 1$  so that

$$\frac{1}{\bar{C}} \leq u_i(t) \leq \tilde{C}, \quad (20)$$

on  $M \times [-(T - t_i)^{-1}t_i, 1/2]$ . Functions  $u_i(t)$  satisfy backward parabolic equations

$$\frac{d}{dt}u_i(t) = -\Delta u_i(t) + R(g_i(t))u_i(t).$$

Uniform  $C^0$  estimate (20) imply,

$$\int_{-1}^{1/2} \int_M |\nabla u_i|^2 dV_{g_i(t)} dt \leq C.$$

Since  $|\nabla u_i| = (4\pi(1 - t))^{-n} |\nabla f_i| u_i$  and (20), we have,

$$\int_{-1}^{1/2} \int_M |\nabla f_i|^2 dV_{g_i(t)} dt \leq \tilde{C}.$$

By standard parabolic estimates applied to  $f_i$  satisfying (3), we have,

$$\sup_{t \in [-1, 1/2]} |f_i(t)|_{C^k} \leq C(k). \quad (21)$$

**Claim 7.** *There is a uniform constant  $C$  so that  $\sup_{t \in [0, T)} \mathcal{W}(g(t), f(t), T - t) \leq C$ .*

*Proof.* Consider our sequence  $t_i \rightarrow T$ . Then, by the scale invariance of  $\mathcal{W}$  and by estimates (21),

$$\begin{aligned} \mathcal{W}(g(t_i), f(t_i), T - t_i) &= \mathcal{W}(g_i(0), f_i(0), 1) \\ &= (4\pi)^{-n} \int_M (R_i + |\nabla f_i|^2 + f_i - 2n) e^{-f_i} dV_{g_i(0)} \\ &\leq C. \end{aligned} \tag{22}$$

By Perelman's monotonicity formula,  $\mathcal{W}(g(t), f(t), T - t)$  increases in time, which together with (22) imply the statement of the claim.  $\square$

The previous claim and Perelman's monotonicity formula for  $\mathcal{W}$  yield the existence of a finite limit,  $\lim_{t \rightarrow T} \mathcal{W}(g(t), f(t), T - t)$ . Let  $a_i = \frac{T+t_i}{2}$ . From before, we have that  $g_i(t) \rightarrow h(t)$ . From our estimates (21) on  $f_i(s)$ , by extracting a subsequence we may assume  $f_i(s) \xrightarrow{C^k(M \times [-1, 1/2])} f_h(s)$ . We also have,

$$\begin{aligned} \mathcal{W}(g(a_i), f(a_i), T - a_i) - \mathcal{W}(g(t_i), f(t_i), T - t_i) &= \int_{t_i}^{a_i} \frac{d}{dt} \mathcal{W} dt = \tag{23} \\ &= \int_{t_i}^{a_i} (4\pi(T - t))^{-n} \int_M (2(T - t) |R_{p\bar{q}} + \nabla_p \nabla_{\bar{q}} f - g_{p\bar{q}}|^2 e^{-f} dV_{g(t)} dt + \\ &+ |\nabla_p \nabla_{\bar{q}} f|^2 + |\nabla_{\bar{p}} \nabla_{\bar{q}} f|^2) e^{-f} dV_{g(t)} dt \\ &\geq (4\pi(T - t_i))^{-n} \int_{t_i}^{a_i} \int_M ((T - t_i) (|R_{p\bar{q}} + \nabla_p \nabla_{\bar{q}} f - g_{p\bar{q}}|^2 + |\nabla_p \nabla_{\bar{q}} f|^2 + \\ &+ |\nabla_{\bar{p}} \nabla_{\bar{q}} f|^2) e^{-f} dV_{g(t)} dt \\ &= (4\pi)^{-n} \int_0^{1/2} \int_M (|\text{Ric}_i + \nabla \bar{\nabla} f_i - g_i|^2 + |\nabla \nabla f_i|^2 + |\bar{\nabla} \bar{\nabla} f_i|^2) e^{-f_i} dV_{g_i(s)} ds. \end{aligned}$$

The left hand side of (23) converges to zero, while its right hand side converges to

$$\int_0^{1/2} \int_M (|\text{Ric}(h) + \nabla \bar{\nabla} f_h - h|^2 + |\nabla \nabla f_h|^2 + |\bar{\nabla} \bar{\nabla} f_h|^2) e^{-f_h} dV_{h(s)} ds.$$

This yields  $h(s)$  is a Kähler Ricci soliton and it satisfies,

$$\begin{aligned} R_{p\bar{q}}(h) + \nabla_p \nabla_{\bar{q}} f_h - h_{p\bar{q}} &= 0, \\ f_{\bar{p}\bar{q}} = f_{pq} &= 0. \end{aligned}$$

In other words, what we get is the following: if we have two different solutions  $u_1 = (4\pi(T-t))^{-n}e^{-f_1}$  and  $u_2 = (4\pi(T-t))^{-n}e^{-f_2}$ , to each of them we can apply the reasoning from above. We can consider  $u_i^1(t) = (T-t_i)^{-n}u_1(t_i+t(T-t_i))$  and  $u_i^2(t) = (T-t_i)^{-n}u_2(t_i+t(T-t_i))$  and as above, we can conclude  $f_1(t_i+t(T-t_i)) \xrightarrow{C^k} f_h^1$  and  $f_2(t_i+t(T-t_i)) \xrightarrow{C^k} f_h^2$ , where  $f_h^1$  and  $f_h^2$  both satisfy,

$$\text{Ric}(h) + \nabla \bar{\nabla} f_h^1 - h = 0,$$

$$\text{Ric}(h) + \nabla \bar{\nabla} f_h^2 - h = 0.$$

This implies  $\Delta f_h^1 = \Delta f_h^2$ , which yields  $f_h^1 = f_h^2 + C$ , for some constant  $C$ . Since  $\int_M e^{-f_h^1} dV_h = \int_M e^{-f_h^2} dV_h$ , we get  $C = 0$ . This means, for  $t \in [-1, 1/2]$ ,

$$\frac{u_1(t_i+t(T-t_i))}{u_2(t_i+t(T-t_i))} = \frac{u_i^1(t)}{u_i^2(t)} \xrightarrow{C^k} 1,$$

and in particular, by putting  $t = 0$ ,

$$\frac{u_1(t_i)}{u_2(t_i)} = \frac{u_i^1(0)}{u_i^2(0)} \xrightarrow{C^k} 1, \quad (24)$$

as  $i \rightarrow \infty$ , where  $u_i^1(t) = (T-t_i)^n u_1(t_i+t(T-t_i))$ , and similarly for  $u_i^2(t)$ .

A simple computation shows that the evolution equation for  $\frac{u_1}{u_2}$ , since both functions  $u_1$  and  $u_2$  satisfy the conjugate heat equation (2) is

$$\frac{d}{dt} \frac{u_1}{u_2} = -\Delta \frac{u_1}{u_2} - \nabla \ln u_2 \nabla \left( \frac{u_1}{u_2} \right). \quad (25)$$

If there is a time  $t_0 \in [0, T)$  and  $x \in M$  so that  $\frac{u_1(x, t_0)}{u_2(x, t_0)} \geq 1 + \delta$ , with  $\delta > 0$ , then  $\max_M(\frac{u_1(\cdot, t_0)}{u_2(\cdot, t_0)}) \geq 1 + \delta$ . By the maximum principle applied to (25), we get  $\max_M(\frac{u_1(\cdot, t)}{u_2(\cdot, t)})$  increases in time and therefore,

$$\max_M \left( \frac{u_1(\cdot, t)}{u_2(\cdot, t)} \right) \geq 1 + \delta,$$

for all  $t \in [t_0, T)$ . This contradicts (24) and henceforth  $u_1(t) = u_2(t)$  for all  $t \in [0, T)$ .  $\square$

*Proof of Theorem 2.* The proof of the theorem follows from Lemma 5, Proposition 3 and Proposition 6.  $\square$

### 3 More on the uniqueness of $u$ for $n = 2$

In this section we will consider two dimensional Kähler, compact manifolds  $M$ , with  $c_1 > 0$ . Let  $g(t)$  be the Kähler Ricci flow on such a manifold. In the Kähler case, the curvature integral  $\int_M |\text{Rm}|^2 dV$  is always bounded in terms of topological invariants, the first and the second Chern class. This integral is scale invariant for  $n = 2$  which implies its significant importance in that case. For example, using that in [9] the following result has been proved.

**Theorem 8.** *Let  $g(t)$  be the normalized Kähler Ricci flow on a manifold as above, with uniformly bounded Ricci curvatures along the flow. Then for every sequence  $t_i \rightarrow \infty$ , there is a subsequence, so that  $(M, g(t_i + t)) \rightarrow (M_\infty, g_\infty(t))$ , where  $M_\infty$  is the orbifold with finitely many isolated singularities and  $g_\infty(t)$  is a singular metric that satisfies the Kähler Ricci soliton equation outside those singular points.*

Combining Theorem 8 and Theorem 2 yields the following result.

**Theorem 9.** *If  $g(t)$  is the unnormalized Kähler Ricci flow on a manifold  $M$  as above, such that  $|\text{Ric}(g(t))| \leq \frac{C}{T-t}$ , for a uniform constant  $C$ , there is a unique solution of the conjugate heat equation (2).*

*Proof.* The proof is analogous to the proof of Theorem 2, since the only singularities we get in two dimensional case are just isolated points. Adopt the notation from the proof of Theorem 2. For the rescaled sequence of metrics  $g_i(t) = (T - t_i)^{-1}g(t_i + t(T - t_i))$ , we have that

$$\sup_{M \times [-t_i(T-t_i)^{-1}, 1]} |\text{Ric}(g_i(t))| \leq C,$$

$$\delta < u_i(t) \leq C,$$

for all  $t \in [-t_i(T - t_i)^{-1}, \frac{1}{2}]$  and therefore,

$$\int_{-1}^1 \frac{1}{2} \int_M |\nabla f_i(s)|^2 dV_{g_i(s)} ds \leq C.$$

The last estimate implies that for every  $i$ , there is  $s_i \in [-1, 1/2]$ , so that

$$\int_M |\nabla f_i(s_i)|^2 dV_{g_i(s_i)} \leq C.$$

In the proof of Claim 7, to prove the boundness of  $\mathcal{W}$ , instead of considering  $\mathcal{W}(g(t_i), f(t_i), T - t_i)$  we will consider  $\mathcal{W}(g(t_i + s_i(T - t_i)), f(t_i + s_i(T - t_i)), T - t_i)$

$t_i))$ ,  $(T - t_i)(1 - s_i))$ , for  $s_i \in [-1, 1/2]$ . The rest of the proof is same. We also have the same estimate (23) as before, where the left hand side tends to zero as  $i \rightarrow \infty$ , due to the monotonicity and the boundness of  $\mathcal{W}$ . Assume  $p_1, \dots, p_N$  are the singular points we get by taking the limit of the sequence  $(M, g(t_i + t))$ , and that  $p_1^i, \dots, p_N^i$  are the curvature concentration points that are responsible for obtaining our singularities in the limit. Let  $\{D_j\}$  be the compact exhaustion of  $M_\infty \setminus \{p_1, \dots, p_N\}$ . Our geometries  $g(t)$  are uniformly bounded on each of  $D_j$ , (those bounds deteriorate when  $j \rightarrow \infty$ , that is, when we approach singularities). Henceforth, the estimate (23) tells us we can extract a subsequence, such that  $(M, g(t_i + t)) \rightarrow (M_\infty, h(t))$  and  $h(t)$  satisfies the Kähler Ricci soliton equation,

$$\text{Ric}(h(t)) + \nabla \bar{\nabla} f_h(t) - h(t) = 0, \quad (26)$$

away from singular points. As in Proposition 6, if we assume there are at least two different solutions of the conjugate heat equation, we will get at least two different functions  $f_h(t)$  and  $f'_h(t)$ , that satisfy (26) away from singular points. Without loss of generality assume there is only one singular point  $p$ .

**Claim 10.** *Functions  $f_h(t)$  and  $f'_h(t)$  coincide on  $M_\infty \setminus \{p\}$*

*Proof.* Choose a sequence  $\eta_k \rightarrow 1$  on  $M_\infty$  with  $\int |\nabla \eta_k|^2 \rightarrow 0$  as  $k \rightarrow \infty$ , e.g.,

$$\eta_k(t) = \begin{cases} 0, & \text{for } x \in B(p, 1/k^2), \\ 1 - \frac{\ln(k^2 \text{dist}_h(p, x))}{\ln k}, & \text{for } x \in B(p, 1/k) \setminus B(p, 1/k^2), \\ 1, & \text{for } x \in M_\infty \setminus B(p, 1/k). \end{cases}$$

Denote by  $F = f_h - f'_h$ . It satisfies,  $\Delta_h F = 0$  away from  $p$ . Multiply it by  $F^2 \eta_k^2$  and then integrate over  $M$ . We get,

$$\begin{aligned} \int |\nabla F|^2 \eta_k^2 dV_h &= - \int \nabla \eta_k \eta_k F \nabla F dV_h \\ &\leq 4 \int |F|^2 |\nabla \eta_k|^2 dV_h + \frac{1}{4} \int \eta_k^2 |\nabla F|^2 dV_h, \\ &\leq C \int |\nabla \eta_k|^2 dV_h + \frac{1}{4} \int \eta_k^2 |\nabla F|^2 dV_h \end{aligned}$$

which after taking  $k \rightarrow \infty$  implies,

$$\int_{M_\infty \setminus \{p\}} |\nabla F|^2 dV_h = 0.$$

As in [1], [2] and [10] one can show  $M_\infty \setminus \{p\}$  is connected. This amounts to having  $F = \text{const}$  on  $M_\infty \setminus \{p\}$ . Because of the integral normalization condition for  $f_h$  and  $f'_h$ , we have  $C = 0$  and therefore,  $f_h = f'_h$ .  $\square$

The rest of the proof of Theorem 9 is as in the proof of Theorem 2.  $\square$

## 4 Reduced distance function with the base point $(p, T)$

In [6] Perelman has introduced the reduced distance function for the Ricci flow  $g(t)$  defined for  $t \in [0, T']$ , with respect to the base point  $(p, T')$ , for some  $p \in M$ , as follows. For any point  $q \in M$  and any  $t \in [0, T']$ , let

$$l(q, T' - t) = \frac{1}{2\sqrt{T' - t}} \int_t^{T'} \sqrt{T' - u} (R(\gamma(u), u) + |\dot{\gamma}|^2) du, \quad (27)$$

where  $\gamma$  is the  $\mathcal{L}$ -geodesic (minimizing the integral in (27)), such that  $\gamma(t) = q$  and  $\gamma(T') = p$ . If in (5) we choose  $\gamma$  to be the  $\mathcal{L}$  geodesic, integrating (5) in  $t$ , if  $\lim_{s \rightarrow T'} \sqrt{T' - s} f(p, s)$  (which is true if  $u = (4\pi(T' - t))^{-n} e^{-f}$  tends to a  $\delta$ -function concentrated at  $p$ , as  $t \rightarrow T'$ ) yields,

$$f(q, t) \leq l(q, T' - t). \quad (28)$$

We would like to define some notion of the reduced distance for the Kähler Ricci flow, defined with respect to the base point  $(p, T)$ , where  $p$  is a point to which our flow shrinks at singular time  $T$ . The idea is roughly as follows. Let  $t_i \uparrow T$  as  $i \rightarrow \infty$ . Fix points  $x, q \in M$  and  $t \in [0, T)$ ; and for each  $i$  such that  $t < t_i$ , define the  $\mathcal{L}$ -distance with the base point  $(x, t_i)$ , from  $(x, t_i)$  to  $(q, t)$  (as Perelman did in [6]). We will denote it by  $L_i^x(q, t)$ . Define

$$\tilde{L}_i(q, t) = \inf_{x \in M} L_i^x(q, t),$$

where the infimum is taken over all base points  $(x, t_i)$ . Assume  $t_i \leq t_j$ . Take any base point  $(x, t_j)$  and let  $\gamma_1$  be a restriction of  $\gamma$  to time interval  $[t, t_i]$ . Then,

$$\begin{aligned} L_j^x(q, t) &= \int_t^{t_j} \sqrt{t_j - u} (R + |\dot{\gamma}|^2) du \geq \int_t^{t_i} \sqrt{t_i - t} (R + |\dot{\gamma}_1|^2) du \\ &\geq \tilde{L}_i(q, t), \end{aligned}$$

If we take the infimum over all base points  $(x, t_j)$  in the previous inequality, we get

$$\tilde{L}_j(q, t) \geq \tilde{L}_i(q, t), \quad (29)$$

which means  $\tilde{L}_i$  is an increasing sequence.

$$\tilde{L}_i(t, q) \leq L_i^q(t, q) \leq \int_t^{t_i} \sqrt{t_i - u} (R + |\dot{\gamma}|^2) du,$$

where we can take  $\gamma(t)$  to be a constant curve  $\gamma(t) = q$ . Then  $\dot{\gamma} = 0$ . Due to Perelman, we have that for the Kähler Ricci flow  $\sup_{M \times [0, T)} |R|(T - t) \leq C$  and therefore,

$$\begin{aligned} \tilde{L}_i(t, q) &\leq \int_t^{t_i} \sqrt{t_i - u} R du \\ &\leq \int_t^{t_i} \sqrt{t_i - u} \frac{C}{T - u} du \leq \int_t^{t_i} \frac{1}{\sqrt{t_i - u}} du \\ &= C\sqrt{t_i - t}. \end{aligned} \quad (30)$$

By (29) and (30) we get there is a  $\lim_{i \rightarrow \infty} \tilde{L}_i(q, t) = \tilde{L}(q, t)$ . This implies  $\tilde{l}_i = \frac{1}{\sqrt{t_i - t}} \tilde{L}_i \rightarrow \frac{1}{\sqrt{T - t}} \tilde{L} = \tilde{l}$ . An estimate (30) implies

$$\sup_{M \times [0, T)} \tilde{l}(q, t) \leq C,$$

for a uniform constant  $C$ .

One interesting question would be whether  $\tilde{l}_i$  satisfy the similar inequalities to those that are satisfied by each of  $l_i^x$ . Recall that Perelman has proved  $l_i^x$  satisfies,

$$-(l_i^x)_t - \Delta l_i^x + |\nabla l_i^x|^2 - R + \frac{n}{t_i - t} \geq 0,$$

$$2\Delta l_i^x - |\nabla l_i^x|^2 + R + \frac{l_i^x - 2n}{t_i - t} \leq 0.$$

**Question:** Do the above inequalities persist after taking the infimum of  $l_i^x$  over all  $x \in M$ ?

If the answer to the above question were positive, this would yield the monotonicity formula for  $\tilde{V}(q) = (T - t)^{-n} \int_M e^{-\tilde{l}} dV_g$ , for all  $t \in [0, T)$ .

## References

- [1] M.Anderson: Ricci curvature bounds and Einstein metrics on compact manifolds; Journal of the American Mathematical Society, Volume 2, Number 3 (1989) 455–490.
- [2] S.Bando, A.Kasue, H.Nakajima: On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth; Invent.math. 97 (1989) 313–349.
- [3] H.D.Cao: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds; Invent. math. 81 (1985) 359–372.
- [4] R. Hamilton: The formation of singularities in the Ricci flow, Surveys in Differential Geometry, vol. 2, International Press, Cambridge, MA (1995) 7–136.
- [5] R. Hamilton: A compactness property for solutions of the Ricci flow, Amer. J. Math. 117 (1995) 545–572.
- [6] G. Perelman: The entropy formula for the Ricci flow and its geometric applications; arXiv:math.DG/0211159.
- [7] N.Sesum, G. Tian, X. Wang : Notes on Perelman’s paper.
- [8] N. Sesum: Convergence of the Ricci flow toward a unique soliton, arXiv:math.DG/0405398, to appear in the Communications in Analysis and Geometry.
- [9] N. Sesum: Convergence of a Kähler-Ricci flow, arXiv:math.DG/0402238, to appear in Mathematical Research Letters.
- [10] G.Tian: On Calabi’s conjecture for complex surfaces with positive first Chern class; Inventiones math. 101 (1990), 101-172.
- [11] R.Ye: Notes on the reduced volume and asymptotic Ricci solitons of  $\kappa$ -solutions (available at <http://www.math.lsa.umich.edu/research/ricciflow/perelman.html>).